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LETTER TO THE EDITOR

Kähler fields and five-dimensional Kaluza–Klein theory

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Abstract. The Einstein–Kähler equations on a five-dimensional pseudo-Riemannian manifold with Kaluza–Klein symmetry are discussed.

The Kähler equation (Kähler 1962, Graf 1978) has recently attracted the attention of physicists (Benn and Tucker 1982b, 1983, Banks *et al* 1983, Becker and Joos 1982). This equation for an inhomogeneous differential form, that is a general element of the exterior algebra generated by a basis for the cotangent space of the space–time manifold, offers, in Minkowski space, an alternative description of half-integral spin to that provided by the conventional formulation of the Dirac equation. It has been suggested (Benn and Tucker 1983, Banks *et al* 1983) that this description of fermions might provide a starting point for their incorporation into Kaluza–Klein theories. In this letter we report on the equations obtained from the Einstein–Kähler action in five dimensions when the metric has Kaluza–Klein symmetry. Our conventions for the Kähler equation are found in Benn and Tucker (1982b, 1983) and full details of the dimensional reduction of the Einstein equation may be found in Benn and Tucker (1982a).

We take the action-density five-form

$$\Lambda = \mathbb{R}_{AB} \wedge \# (E^A \wedge E^B) + \lambda S_0\{\bar{\Phi}_\vee E^A_\vee \nabla_{X_A}\Phi + m\bar{\Phi}_\vee \Phi\} Z \tag{1}$$

where E^A, $\#$ and $\mathbb{R}_{AB}$ are the orthonormal coframes, Hodge map and curvature two-forms associated with the five-dimensional metric. ∇_{X_A} denotes the pseudo-Riemannian covariant derivative with respect to a vector of an orthonormal frame, and $\vee$ is the Clifford product defined with respect to the five-dimensional metric. λ and m are arbitrary constants and Z is the volume five-form $\# 1$. S_p projects p-form components from an arbitrary element of the exterior algebra. Φ is a real Kähler field, that is an element of the (real) exterior algebra, and

$$\bar{\Phi} \equiv \xi\eta\Phi \tag{2}$$

where the automorphism η and anti-automorphism ξ commute with S_p and satisfy

$$\eta S_p\Phi = (-1)^p S_p\Phi \tag{3}$$

$$\xi S_p\Phi = (-1)^{[p/2]} S_p\Phi \tag{4}$$

where $[p/2]$ is the integer part of $p/2$. The five-dimensional field equations obtained from (1) are

$$\mathbb{R}_{AB} \wedge \# (E^A \wedge E^B \wedge E_C) + \lambda T_{CA} \# E^A = 0 \tag{5}$$

$$E^A{}_\vee \nabla_{X_A}\Phi + m\Phi = 0 \tag{6}$$

where

$$T_{CA} = \tfrac{1}{2}S_0\{(E_{c\vee}\Phi_\vee E_A + E_{A\vee}\Phi_\vee E_c)_\vee E^D{}_\vee \nabla_{X_D}\bar{\Phi}\} \tag{7}$$

and capital indices are raised and lowered with the matrix of orthonormal five-metric components $\eta_{AB} = \mathrm{diag}(-1, 1, 1, 1, 1)$. It should be noted that the Kähler action only involves the pseudo-Riemannian connection (in fact it can be written solely in terms of the Hodge map and the exterior derivative), and thus, in constrast to the usual Cartan–Dirac formulation, these equations involve no torsion.

To analyse these equations we assume orthonormal co-frames of the form

$$E^a = e^a \qquad a = 0, \ldots, 3 \tag{8}$$

$$E^5 = \rho(\mathrm{d}x^5 + A) \tag{9}$$

where the e^a are one-forms on four dimensions and are identified as orthonormal with respect to the Lorentzian space–time metric. ρ is a Jordon–Thirry field taken for simplicity to satisfy

$$\partial\rho/\partial x^5 = 0 \tag{10}$$

and A may be interpreted as the electromagnetic potential one-form on space–time. The Einstein equations (5) then reduce to

$$R_{ab} \wedge *(e^a \wedge e^b \wedge e_c) + \tfrac{1}{2}\rho^2(i_c \wedge F \wedge *F - F \wedge i_c * F) + (2/\rho)Di_c * \mathrm{d}\rho + \lambda T_{ca} * e^a = 0 \tag{11}$$

$$\mathrm{d}(\rho^3 * F) = \lambda\rho^2 T_{5a} * e^a \tag{12}$$

$$\mathrm{d} * \mathrm{d}\rho = \tfrac{1}{2}\rho^3 F \wedge *F - \tfrac{1}{3}\lambda\rho(T_{55} - \tfrac{1}{2}T^a{}_a) * 1. \tag{13}$$

R_{ab} are the curvature two-forms and $*$ the Hodge map associated with the space–time metric. D is the SO(3, 1) space–time covariant exterior derivative and $F = \mathrm{d}A$. We decompose Φ as

$$\Phi = \tfrac{1}{2}(\alpha + \beta) + \tfrac{1}{2}(\alpha - \beta) \wedge E^5 \tag{14}$$

where α and β are x^5-dependent Kähler fields on space–time. Equation (6) then becomes

$$\begin{aligned} &e^a{}_\vee \nabla_{X_a}\alpha - A_\vee \frac{\partial}{\partial x^5}\alpha + \frac{\rho}{4}(F_\vee\eta\alpha + \eta\alpha_\vee F) - \frac{\rho}{2}(F_\vee\eta\beta + i_a F_\vee i^a\eta\beta) \\ &\quad + \frac{1}{2\rho}\mathrm{d}\rho_\vee\alpha - \frac{1}{2\rho}\eta\beta_\vee \mathrm{d}\rho + \frac{1}{\rho}\frac{\partial(\eta\alpha)}{\partial x^5} + m\alpha = 0 \end{aligned} \tag{15}$$

$$\begin{aligned} &e^a{}_\vee \nabla_{X_a}\beta - A_\vee \frac{\partial}{\partial x^5}\beta - \frac{\rho}{4}(F_\vee\eta\beta + \eta\beta_\vee F) + \frac{\rho}{2}(F_\vee\eta\alpha + i_a F_\vee i^a\eta\alpha) \\ &\quad + \frac{1}{2\rho}\mathrm{d}\rho_\vee\beta - \frac{1}{2\rho}\eta\alpha_\vee \mathrm{d}\rho - \frac{1}{\rho}\frac{\partial}{\partial x^5}(\eta\beta) + m\beta = 0. \end{aligned} \tag{16}$$

∇ denotes the space–time pseudo-Riemannian connection and the $\{X_a\}$ are an orthonormal basis of vectors on space–time dual to the $\{e^a\}$. The Clifford product associated with the space–time metric is denoted $\vee$. By $\partial\alpha/\partial x^5$ we mean the inhomogeneous form on space–time whose components are the partial derivatives with respect to x^5 of those of α.

To complete the reduction the components of α and β may be expanded as Fourier series in x^5. It is then seen that harmonics of α and β have masses proportional to their charge, a feature in common with the Dirac equation reduced from five dimensions (Souriau 1963, Thirring 1973, Dereli and Tucker 1982). Also in common with the reduced Dirac equation are the Pauli interactions $F_\vee \eta\alpha$ as well as the minimal interactions $A_\vee\, \partial\alpha/\partial x^5$. The most obvious difference the Kähler equation presents is that one Kähler field in five dimensions reduces to two in four dimensions, whereas if we take a Dirac equation for a spinor in five dimensions then we may reduce this to an equation for a four-dimensional spinor. Each of the fields α and β is equivalent to four Dirac spinors, which will in general be coupled unless the four-dimensional space–time is flat. In addition, we now have explicit interactions occurring as right Clifford multiplication, e.g. $\eta\alpha_\vee F$. These terms will split the degeneracy of the four Dirac spinors contained in α even if the four-dimensional space–time may be approximate by Minkowski space. Further, there are terms coupling the α and β equations. It is thus seen that just as four-dimensional curvature couples the four different spinors in the four-dimensional Kähler equation, so does electromagnetism when it is a facet of the curvature of a five-dimensional Kaluza–Klein manifold.

References

Banks T, Dothan Y and Horn D 1983 *Phys. Lett.* **117B** 413
Becker P and Joos H 1982 *The Dirac–Kähler Equation and Fermions on the Lattice, DESY 82-031*
Benn I M and Tucker R W 1982a *Phys. Lett.* **112B** 344
—— 1982b *Phys. Lett.* B **119** 348
—— 1983 *Fermions without spinors, Lancaster University preprint, Commun. Math. Phys.* to be published
Dereli T and Tucker R W 1982 *Nucl. Phys.* B **209** 217
Graf W 1978 *Ann. Inst. Henri Poincaré* **XXIX** 85
Kähler E 1962 *Rendiconti di Matematics* (3–4) **21** 425
Souriau J-M 1963 *Nuovo Cimento* **30** 565
Thirring W 1973 in *The Physicist's Conception of Nature* ed J Mehra (Boston: Beidel) pp 199–201